INERTIAL OSCILLATIONS INDUCED BY PROPAGATION OF VISTULA LAGOON FRESHWATER INTO THE BALTIC SEA

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Abstract: A mathematical model of the mechanism of the appearance of antisymmetric vortices during the propagation of freshwater into the seawater which is observed, in particular, at the exit from the Baltic Canal connecting the Vistula Lagoon and the Baltic Sea is constructed in the work. In particular it is shown that the main reason for the vortex formation in this case is the Coriolis force. The exact dependence of the circulation of velocity on time for the three simplest types of the "tongue" of the intrusion of freshwater is calculated analytically in the work as well.

Keywords: Cyanobacteria, Vistula Lagoon, Baltic Strait, Coriolis force, vorticity

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1. Introduction

One of the most noteworthy results of the numerical simulation of the mixing process of two water masses with different characteristics (such as density, salinity, temperature, *etc.*) is the description of the process of the inflow of a tongue of freshwater into a larger salt water reservoir [1–3]. In particular, these models unanimously predict the emergence of two asymmetrical vortices in the process of mixing, which effect occurs in full agreement with the numerous in-situ data (see, for example, [4–7]). The purpose of this article is to develop an exact mathematical theory describing this phenomenon which can be used to establish the features of the behavior of water masses in the vicinity of the canal connecting the Baltic Sea and the Vistula Lagoon (see Figure 1).

One of the basic mathematical tools for studying vortices is the Björknes theorem which provides the necessary and sufficient conditions for vortex generation [8]. Unfortunately, this theorem in the classical form is applicable only to water masses moving under the action of conservative forces. Nonetheless, it is

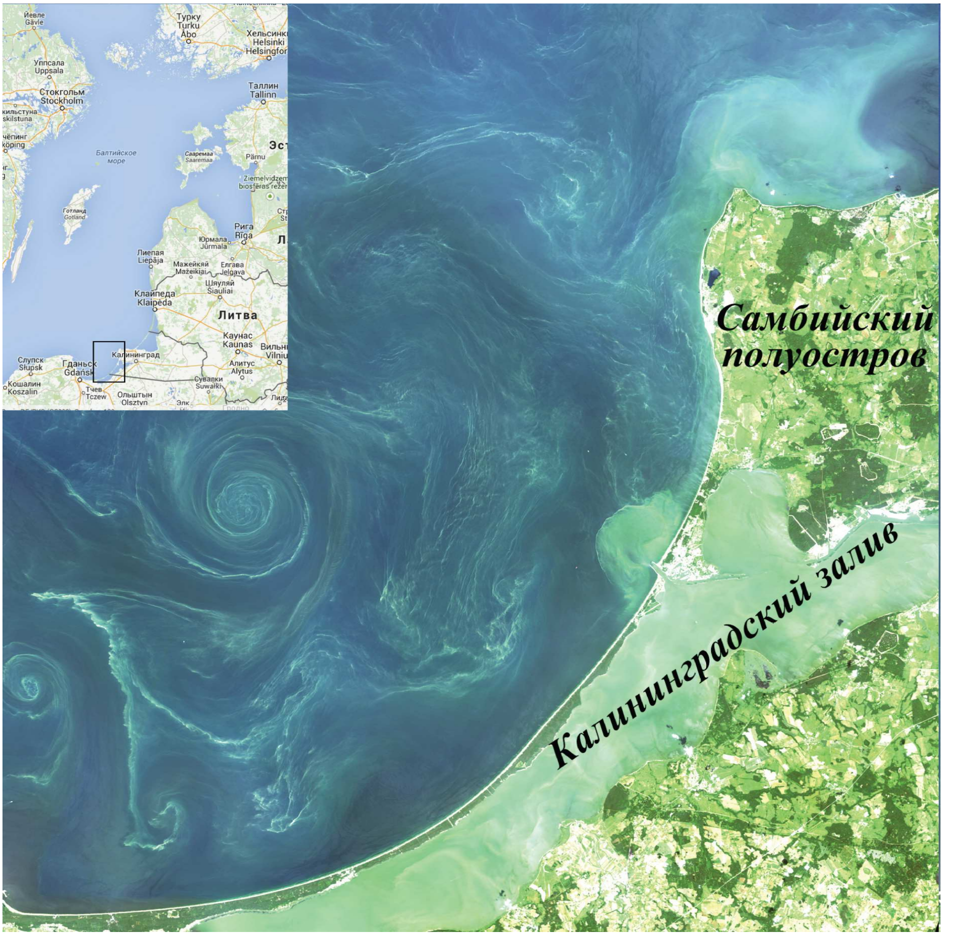

Figure 1. The plume of water inflow from the Kaliningrad (Vistula) Lagoon. We pay special attention to the shape of this plume, which unambiguously evidences the presence of at least two oppositely directed vortices. Photo of 07/07/2014 taken with using the Operational Land Imager from the remote sensing satellite LandSat-8

necessary to take into account such a non-conservative force as the Coriolis force to correctly describe such a problem as the modeling of the process of Vistula Lagoon water inflow into the Baltic Sea. Thus, we come to the conclusion that it is necessary to modify the Björknes theorem for the problem described in order to take the rotation of the Earth into account.

2. Vorticity, isobaric-isosteric tubes and the modified Björknes Theorem

A key role in our discussions will be played by the notion of velocity circulation in the environment. We define it as follows. Let $\vec{v}=\{v_x,v_y,v_z\}$ – the vector field of velocities in a given volume of a fluid, and let L – be a simple smooth closed contour lying entirely within this volume. Then, the circulation of the velocity along the contour L is given by the formula

$$\gamma \equiv \oint_L \overrightarrow{ds}\vec{v} \tag{1}$$

We will be interested in evolution of γ with time, *i.e.* the quantity $d\gamma/dt$. To calculate this derivative, we introduce a new angular variable μ, such that the integration contour could be represented in the parametric form:

$$\vec{r}=\vec{r}\,(\mu,t), \qquad d\vec{s}=\frac{\partial\vec{r}}{\partial\mu}d\mu \tag{2}$$

For clarity, we will assume that the parameter μ varies within the range $[0,2\pi]$. Differentiating (1) with respect to time and taking into account that t and μ are independent variables, we obtain:

$$\frac{d\gamma}{dt}=\oint_L d\vec{s}\frac{d\vec{v}}{dt}+\int_0^{2\pi} d\mu\vec{v}\frac{\partial^2\vec{r}}{\partial t\partial\mu} \tag{3}$$

It is not difficult to see that the second integral contains the total derivative of the square of the velocity $\vec{v}$ by μ. This means that the second term in (3) is equal to:

$$\frac{1}{2}\left(\vec{v}^2\,(2\pi)-\vec{v}^2\,(0)\right)=0 \tag{4}$$

from which comes the important formula

$$\frac{d\gamma}{dt}=\frac{d}{dt}\oint_L d\vec{s}\vec{v}=\oint_L d\vec{s}\frac{d\vec{v}}{dt} \tag{5}$$

In the next step, we need an equation for the motion of the fluid, which, neglecting the effects of viscosity, has the form:

$$\frac{d\vec{v}}{dt}=-\frac{\nabla p}{\rho}+\vec{F} \tag{6}$$

where p – pressure, ρ – density, and $\vec{F}$ – external force. Let us assume at the beginning that the force $\vec{F}$ – is conservative, *i.e.* that there exists such a scalar function $\varphi=\varphi(\vec{x},t)$, called the potential that $\vec{F}=\nabla\varphi$. In this case, the integral along the contour L of the function $\vec{F}$ turns to zero, since:

$$\oint_L \vec{F}d\vec{s}=\oint_L \nabla\varphi d\vec{s}=\oint_L d\varphi=0 \tag{7}$$

Taking into account (7), with the direct substitution of (6) into (5) we obtain the following formula

$$\frac{d\gamma}{dt}=-\oint_L \frac{\nabla p}{\rho}d\vec{s}=-\oint_L \rho^{-1}dp=-\oint_L \omega dp \tag{8}$$

Where we introduced a new function $\omega=\rho^{-1}$, meaning the volume of the fluid per unit of mass. We note that it follows directly from the formula (8) that in incompressible fluids for which $\rho = \text{const}$, deriving $d\gamma/dt = 0$, and the vorticity turns out a quantity independent of time. A similar conclusion can be drawn for a more general class of barotropic fluids, *i.e.* fluids for which $p = p(\rho)$, as for these fluids, the integrand in (8) can also be represented as the gradient of the following scalar function ψ:

$$\psi(\rho) = -\int \rho^{-1} dp(\rho) \tag{9}$$

which means that the integral along the closed contour L in (8) should be equal to zero.

Let us assume now that the fluid under study does not satisfy the barotropic condition. Let the values of the pressure and the specific volume (*i.e.* the quantities of reciprocal density) at a certain point in space be p_0 and ω_0, respectively. We will consider two isobaric surfaces, on one of which $p = p_0 = const$, and on the other $p = p_0 + 1$ and two isosteric surfaces with specific volume values $\omega = \omega_0$ and $\omega = \omega_0 + 1$. The intersection of these four surfaces forms the so-called *single isobaric-isosteric tube*. It is easy to verify that the contour integral (8) over this tube is equal to ± 1, depending on the selected direction of the traversal.[1] Depending on the sign, positive and negative single isobaric-isosteric tubes are distinguished. If the contour L is selected in this case in such a way that it should include an integer number of both isobaric and isosteric surfaces, then (8) simply transforms into:

$$\frac{d\gamma}{dt} = N_1 - N_2 \tag{10}$$

where N_1 and N_2 – the total number of positive and negative tubes. Thus, we come to the conclusion that the vorticity changes (occur in this sense), if the isobaric and isosteric surfaces *do not coincide*. This is the essence of the Björknes theorem.

In deriving the formula (10), we used the assumption that the total force acting on the fluid was conservative. This assumption was fully confirmed when describing the motion of the nonrotating fluid in the gravitational field. Nevertheless, the situation becomes somewhat complicated as soon as we take into consideration the fact of *the Earth's rotation around its axis*. In this case, the coordinate system associated with a fixed point on the Earth's surface turns out to be non-inertial, which leads to the appearance of two additional non-inertial forces: the Coriolis force $\overrightarrow{F_c}$ and the centrifugal force $\overrightarrow{F_{ct}}$, having the following form:

$$\begin{aligned} \overrightarrow{F_c} &= -2\left[\vec{\Omega}, \vec{v}\right] \\ \overrightarrow{F_{ct}} &= \left[\vec{\Omega}, \left[\vec{r}, \vec{\Omega}\right]\right] \end{aligned} \tag{11}$$

where $\vec{\Omega}$ – the angular velocity vector of the Earth's rotation parallel to its axis of rotation and directed from the south pole to the north pole [9]. We note that

1. It is this direction from the pressure gradient to the specific volume gradient that is usually chosen.

the force $\overrightarrow{F_{ct}}$ is always directed from the axis of rotation and it is conservative. This means that when considering the rotation of the Earth, the equations of motion (6) must be rewritten in the form:

$$\frac{d\vec{v}}{dt} = -\frac{\nabla p}{\rho} - 2\left[\vec{\Omega}, \vec{v}\right] + \nabla U \tag{12}$$

where U – the effective potential energy including centrifugal effects. Repeating our previous considerations for this case, and taking into account (12), we arrive at the following equation:

$$\frac{d\gamma}{dt} = N_1 - N_2 - 2\oint\limits_L \left[\vec{\Omega}, \vec{v}\right] d\vec{s} \tag{13}$$

Thus, the rotation of the Earth leads to the appearance of an additional term in the equation for the rate of change in the velocity circulation. We note that the integrand in (13) implicitly takes into account the geographical latitude θ. Indeed, suppose that at some point on the Earth's surface that lies at the latitude θ a coordinate system is chosen, in which (for definiteness) axis $\vec{x}$ is directed from the east to the west, axis $\vec{y}$ – from the north to the south, and axis $\vec{z}$ is normal to the Earth's surface and is directed away from it. Then, the vector product $\left[\vec{\Omega}, \vec{v}\right]$ will have the form:

$$\left[\vec{\Omega}, \vec{v}\right] = -2|\vec{\Omega}|\left(\left(v_z \cos\theta + v_y \sin\theta\right)\vec{i} - v_x \sin\theta \vec{j} - v_x \cos\theta \vec{k}\right) \tag{14}$$

Finally, in the case when $v_z = 0$ and the contour L is parallel to the plane $0xy$, the formula (13) can be simplified and reduced to the following form:

$$\frac{d\gamma}{dt} = N_1 - N_2 - 2|\vec{\Omega}|\sin\theta \frac{dS}{dt} \tag{15}$$

where $S(t)$ – the area limited by the contour L.

One example of the application of the formula (15) is the problem of the initiation of vortices in the inflow of a freshwater river into the sea mentioned at the beginning of the article. As is easily seen in this case – the isobaric and isosteric surfaces coincide with good accuracy, therefore the conditions of the classical Björknes theorem are not satisfied. The reason that leads to the actually observed appearance of vortices is the rotation of the Earth included in the last term in (15). The point here is that taking viscosity into account leads to a bell-shaped velocity distribution in the fresh stream. For instance, in the examples below, this distribution follows a linear or parabolic law. In any case, the velocity of liquid particles in the central part of the flow is greater than that of the particles near the fresh-saltwater interface, and therefore it is subject to a different effect of the Coriolis force. Consequently, an arbitrary closed contour consisting of some fixed liquid particles of freshwater will deform with time and change its area. This will lead to the "inclusion" of the Coriolis term in (15), and hence – ultimately – it will lead to the appearance of two vortices.

3. Studying the vorticity inside of the Baltiysk strait: the three models

Now, let us proceed to the analysis of a specific model describing the appearance of vorticity when a freshwater river flows into the sea. For simplicity, in the subsequent discussion we assume that the x-component of the velocity is zero, and hence we reduce the problem to the plane case. Also, since the Coriolis force is the only force contributing to (13), we will further assume the remaining forces (gravity, centrifugal, *etc.*) to be zero to simplify the calculations. In addition, we will adopt the following assumptions:

(1) at the initial moment of time, the velocity distribution in the river is symmetrical to the axis y chosen in the middle of the river mouth and has the form:

$$\overrightarrow{v_0} = \{0, v_0(x)\} \tag{16}$$

(2) the circulation along the contour is studied which, with $t=0$, coincides with the square $\{(x,y): x \in [0,L], y \in [0,L]\}$ with side L.

We will consider two variants of the velocity distribution in (16):

$$v_0(x) = a(L-x) \tag{17}$$

$$v_0(x) = a\left(L^2 - x^2\right) \tag{18}$$

where a – some parameter that must be chosen in accordance with phenomenological data.

Let us consider liquid particles on which the Coriolis force acts. According to (12) and (14), the equations of their dynamics have the form:

$$\dot{u} = fv \tag{19}$$

$$\dot{v} = -fu \tag{20}$$

where u denotes the x-component of velocity, v – its component, the parameter $f = 2|\vec{\Omega}|\sin\theta$, and the dot means the total time derivative.

First, we will find the solution of the system (19)–(20) for the general case:

$$\begin{aligned} u(x_0) &= u_0 \\ v(x_0) &= v_0 \end{aligned} \tag{21}$$

For this purpose, it is convenient to multiply (20) by i, add to (19), and introduce a new complex-valued function $\psi = u + iv$. The function ψ should satisfy the following differential equation:

$$\dot{\psi} = -if\psi \tag{22}$$

Having integrated it and having divided the result into real and imaginary parts, we arrive at the following system:

$$\begin{aligned} u &= C_1 \cos(\delta - ft) \\ v &= C_1 \sin(\delta - ft) \end{aligned} \tag{23}$$

where C_1 and δ – constants of integration. The values of these constants can be easily found from the initial conditions (21), leading to the following equations:

$$\begin{aligned} u &= \pm\sqrt{u_0^2+v_0^2}\cos\left(\arctan\frac{v_0}{u_0}-ft\right) \\ v &= \pm\sqrt{u_0^2+v_0^2}\sin\left(\arctan\frac{v_0}{u_0}-ft\right) \end{aligned} \tag{24}$$

The system (24) can be rewritten in a simpler form, if we introduce the following notation:

$$V_0=\sqrt{u_0^2+v_0^2}, \qquad \varphi_0=\arctan\frac{v_0}{u_0} \tag{25}$$

having the meaning of the modulus of the velocity vector $\vec{V}=\{u_0,v_0\}$ and the angle of inclination of this vector to the abscissa axis, respectively. Taking these notations into account, the system (24) becomes

$$u=\pm V_0\cos(\varphi_0-ft), \qquad v=\pm V_0\sin(\varphi_0-ft)$$

Integrating it, we obtain the final solution:

$$\begin{aligned} x &= x_0 \pm \frac{2V_0}{f}\cos(\varphi_0-ft)\sin\left(\frac{ft}{2}\right) \\ y &= y_0 \pm \frac{2V_0}{f}\sin(\varphi_0-ft)\sin\left(\frac{ft}{2}\right) \end{aligned} \tag{26}$$

It follows directly from the (26) in particular that the distance between the body at the moment of time t and its initial position are described by the following expression:[2]

$$r^2=(x-x_0)^2+(y-y_0)^2=\frac{4V_0^2}{f^2}\sin^2\left(\frac{ft}{2}\right) \tag{27}$$

This means that the initially selected contour L undergoes deformation with time (described by the formulas (26)). Hence, according to the modified Bjërknes theorem (15), we inevitably arrive at the necessary and sufficient condition for the vortex formation.

In order to carry out a more rigorous analysis, we need to know the exact form of the evolution of a given curve that with $t=0$

$$y_0=y_0(x_0) \tag{28}$$

For this it is necessary to express x_0 from the first equation of the system (26) and substitute it in the second equation, which will exactly give the unknown law:

$$y=y(x,t) \tag{29}$$

Unfortunately, in an explicit form such a procedure can be performed only in exceptional cases, with a particularly simple definition of the initial velocity

2. We remind the reader that this formula is valid for a body on which only the Coriolis force acts. The discarded conservative forces will obviously give an additional correction to the solution of (26); however, as shown above, this correction inevitably vanishes when integrating along a closed contour in (13).

distribution in (16), in particular, with the choice of (17) or (18). How it is easy to see, in this case $V_0 = v_0$ and $\varphi_0 = \pi/2$, therefore, the system (26) takes the form

$$\begin{aligned} x &= x_0 + \frac{2v_0(x_0)}{f}\sin^2\frac{ft}{2} \\ y &= y_0 + \frac{v_0(x_0)}{f}\sin ft \end{aligned} \tag{30}$$

Example 1. Linear model (17).

Substituting (17) into (30) and choosing for simplicity the plus sign in (30), we obtain

$$x = x_0 + \frac{2a(L - x_0)}{f}\sin^2\frac{ft}{2}, \qquad y = y_0 + \frac{a(L - x_0)}{f}\sin ft \tag{31}$$

whence

$$x_0 = \frac{fx - 2aL\sin^2\frac{ft}{2}}{f - 2a\sin^2\frac{ft}{2}} \tag{32}$$

Consequently, the evolution of the contour (29) is described by the formula:

$$y(x,t) = y_0 + \frac{a(L - x)\sin ft}{f - 2a\sin^2\frac{ft}{2}} \tag{33}$$

Let us consider the motion of the parts of the studied contour separately.

(i) $x_0 = 0$, y – increases. According to (31):

$$x = \frac{2aL}{f}\sin^2\frac{ft}{2}, \qquad y = y_0 + \frac{aL}{f}\sin ft \tag{34}$$

It follows from (34) that the investigated segment is displaced, remaining parallel to the ordinate axis, since the coordinates of all its points are the same at all times.

(ii) Let $y_0(x_0) = L$, x – increases. From (33) we obtain

$$y(x,t) = L + \frac{a(L - x)\sin ft}{f - 2a\sin^2\frac{ft}{2}} \tag{35}$$

This is a straight line with an angular coefficient:

$$\kappa = -\frac{a\sin ft}{f - 2a\sin^2\frac{ft}{2}} \tag{36}$$

(iii) On the segment $x_0 = L$:

$$x = x_0, \qquad y = y_0 \tag{37}$$

Thus, the segment BC stands in place, which was, of course, obvious beforehand, since at the initial moment of time the velocity of all its points was zero, and hence the Coriolis acceleration of these points was zero.

(iv) Finally, on the last segment $y_0(x_0) = 0$, therefore

$$y(x,t) = \frac{a(L - x)\sin ft}{f - 2a\sin^2\frac{ft}{2}} \tag{38}$$

This is a straight line, parallel (35) and shifted down by the quantity L.

Summing up all together, we conclude that at an arbitrary moment of time, the contour that was originally a square with a side L turns into a parallelogram.

To calculate the area of this parallelogram, it is convenient to designate the sides (ii) and (iv) as $y_1(x,t)$ (see (35)) and $y_2(x,t)$ (see (38)), respectively. Then, the area S of the unknown parallelogram is determined by the formula:

$$S = \int_{\left(2aL\sin^2\frac{ft}{2}\right)/f}^{L} dx\,(y_1 - y_2) = L^2\left(1 - \frac{2a}{f}\sin^2\frac{ft}{2}\right) \tag{39}$$

Let $N_1 = N_2$ in (15). Integrating (15) one time we obtain:

$$\gamma(t) = -fS(t)\sin\theta + C \tag{40}$$

We choose the integration constant C so that $\gamma(0) = 0$, whence

$$C = fL^2\sin\theta \tag{41}$$

Substituting in (40) we get the final answer:

$$\gamma(t) = 2aL^2\sin\theta\sin^2\frac{ft}{2} \tag{42}$$

Since our final task is to describe the circulation of the freshwaters of the Vistula Lagoon at the exit from the Strait of Baltiysk (connecting the Vistula Lagoon and the Baltic Sea), we need to substitute in the formula (42) the values of the corresponding parameters a, L, θ and f.

We are going to choose the parameter L based on the size of the mouth of the Strait of Baltiysk: $L \approx 320\,\mathrm{m}$. The geographic latitude θ is $\theta = 54°38'$, therefore $\sin\theta \approx 0.816$. We find the rotation parameter f, starting from the fact that the period of revolution of the Earth around its axis is $T = 24$ hours, and therefore:

$$f = 2\Omega\sin\theta = 2\frac{2\pi}{T}\sin\theta \approx 0.427\,\mathrm{h}^{-1} \tag{43}$$

Finally, to calculate the parameter a, in (17) we assume that $x = 0$ (it corresponds to the middle of the mouth of the strait) and we substitute the value $v_0 = -0.05\,\mathrm{m/s}$, observed at a depth of 4 meters [10]:

$$a = v_0/L \approx -0.563\,\mathrm{h}^{-1} \tag{44}$$

With this in mind, the formula (42) takes the following final form:

$$\gamma_1(t) = -9.4\cdot 10^4 \sin^2(0.21t)\ \mathrm{m}^2/\mathrm{h} \tag{45}$$

It can be seen from this formula, in particular, that the vorticity of the freshwater flow at the outlet from the Strait of Baltiysk is a strictly periodic function and reaches the maximum absolute value $|\gamma_{\max}| = 9.4\cdot 10^4$ in 7 hours 21 minutes after the beginning of observations. A characteristic graph of the behavior of the function $\gamma_1(t)$ is shown in Figure 2.

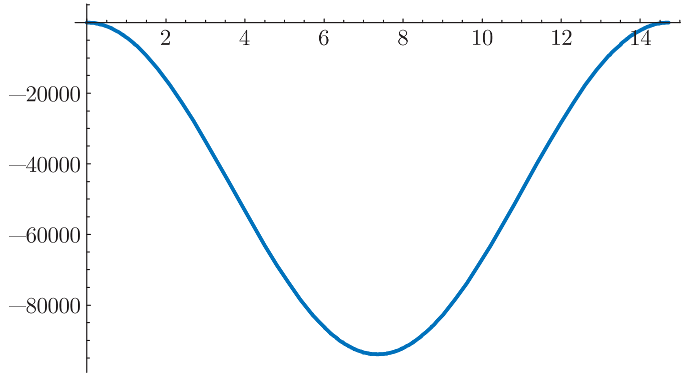

Figure 2. Dependence of vorticity γ on time for the linear model (17)

Example 2. Quadratic model (18). In this case:

$$x = x_0 + \frac{2a\left(L^2 - x_0^2\right)}{f} \sin^2 \frac{ft}{2}, \qquad y = y_0 + \frac{a\left(L^2 - x_0^2\right)}{f} \sin ft \tag{46}$$

We shall proceed similarly to the previous example and consider the evolution of different parts of the contour separately.

(i) $x_0 = 0$, y – increases. We have

$$x = \frac{2aL^2}{f} \sin^2 \frac{ft}{2}, \qquad y = y_0 + \frac{aL^2}{f} \sin ft \tag{47}$$

Obviously, as in Example 1, this segment moves, remaining all the time parallel to the axis y.

(ii) Let $y_0(x_0) = L$, x – increases. Then

$$x = x_0 + \frac{2a\left(L^2 - x_0^2\right)}{f} \sin^2 \frac{ft}{2}, \qquad y = L + \frac{a\left(L^2 - x_0^2\right)}{f} \sin ft \tag{48}$$

Whence

$$x = \pm\sqrt{L^2 - \frac{f(y-L)}{a \sin ft}} + (y-L)\tan\frac{ft}{2} \tag{49}$$

(iii) The segment corresponding to $x_0 = L$ – motionless.

(iv) Let $y_0(x_0) = 0$. This means that

$$x = \pm\sqrt{L^2 - \frac{fy}{a \sin ft}} + y \tan\frac{ft}{2} \tag{50}$$

Thus, the final expression for the area at an arbitrary moment of time takes the form

$$S = L^2 \left(1 - \frac{2aL}{f} \sin^2 \frac{ft}{2}\right) \tag{51}$$

and therefore, according to the formula (15):

$$\gamma(t) = 2aL^3 \sin\theta \sin^2 \frac{ft}{2} \tag{52}$$

We note that, as in the previous example, expression (52) was obtained with the initial condition $\gamma(0) = 0$.

To apply the formula (52) to the problem of inflowing freshwater from the Vistula Lagoon to the Baltic Sea, we will repeat the calculations made in the previous case by replacing the parameter a by the following expression:

$$a = v_0/L^2 \approx -1.76 \cdot 10^{-3} (\mathrm{m} \cdot \mathrm{h})^{-1} \tag{53}$$

as a result of which the formula (45) takes the following form (see also Figure 3):

$$\gamma_2(t) = -294 \cdot \sin^2(0.21t)\ \mathrm{m}^2/\mathrm{h} \tag{54}$$

Having compared this formula with the formula obtained for the linear model (17), we arrive at the following conclusion: although the periodic dependence on time remains low in both models, the change in the form of the "tongue" of freshwater leads to an extremely sharp (more than 300 times!) change of the amplitude of the oscillations of the unknown function. Moreover, even a change of the contour L does not lead to such a sharp change in the amplitude, which can easily be shown in the following example.

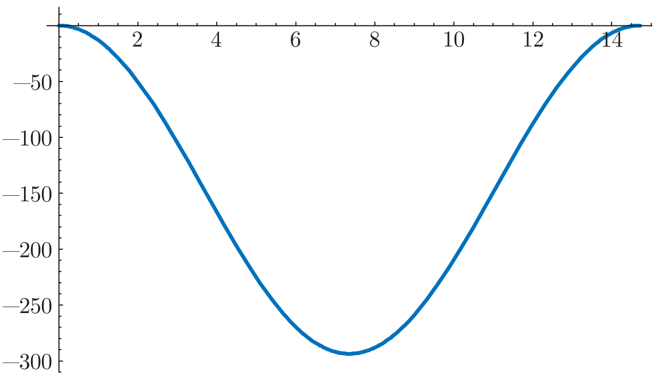

Figure 3. Dependence of vorticity γ on time for the linear quadratic model (18)

Example 3.

Let the initial velocity distribution be of the form (17), but now we choose the contour not as a square, but as a circle of radius R, and we shall assume that $L = 2R$. In this case we obtain

$$\left(y(x,t) - R - \frac{a(2R-x)\sin ft}{f - 2a\sin^2 \frac{ft}{2}} \right)^2 + \left(\frac{fx - 4aR\sin^2 \frac{ft}{2}}{f - 2a\sin^2 \frac{ft}{2}} - R \right)^2 = R^2 \tag{55}$$

The equation (55) determines the dynamics of an initially circular contour through an implicitly defined function $y(x,t)$.

It is convenient to calculate the area as follows. The contour (55) is limited by the two functions:

$$\begin{aligned} y_1 &= R + \mu\left(1 - \frac{x}{2R}\right) - \sqrt{R^2 - (\lambda x - \nu - R)^2} \\ y_2 &= R + \mu\left(1 - \frac{x}{2R}\right) + \sqrt{R^2 - (\lambda x - \nu - R)^2} \end{aligned} \tag{56}$$

where

$$\mu=\frac{2aR\sin ft}{f-2a^2\sin^2\frac{ft}{2}},\qquad \lambda=\frac{f}{f-2a\sin^2\frac{ft}{2}},\qquad \nu=\frac{4aR\sin^2\frac{ft}{2}}{f-2a\sin^2\frac{ft}{2}} \tag{57}$$

Obviously, for $t=0$ the contour will be inside a square with side $2R$. Having performed calculations similar to those made in the previous examples, we come to the conclusion that the area of the deformed contour is

$$S=\int\limits_{4aR\sin^2\frac{ft}{2}/f}^{2R} dx\,(y_2-y_1)=2\int\limits_{4aR\sin^2\frac{ft}{2}/f}^{2R} dx\sqrt{R^2-(\lambda x-\nu-R)^2} \tag{58}$$

As a result of integration, we obtain

$$S(t)=\frac{\pi R^2}{f}\left(f-2a\sin^2\frac{ft}{2}\right)+C \tag{59}$$

Choosing C by analogy to the previous examples, we obtain:

$$\gamma_3(t)=2\pi R^2 a\sin\theta\sin^2\frac{ft}{2} \tag{60}$$

Having repeated the calculations performed for the linear model, we arrive at the following formula:

$$\gamma_3(t)=-7.4\cdot 10^4\sin^2(0.21t)\ \mathrm{m}^2/\mathrm{h} \tag{61}$$

It is easy to see that in the linear model the amplitudes γ_1 and γ_3 changing in the circulation along two different contours are comparable quantities ($9.4\cdot 10^4$ and $7.4\cdot 10^4$), while in the quadratic model the maximum amplitude turns out to be a value not exceeding 0.4% of the amplitudes demonstrated by the linear model. This feature is particularly obvious if the logarithms of the modules of all three functions γ_1, γ_2 and γ_3 are illustrated in one graph – see Figure 4.

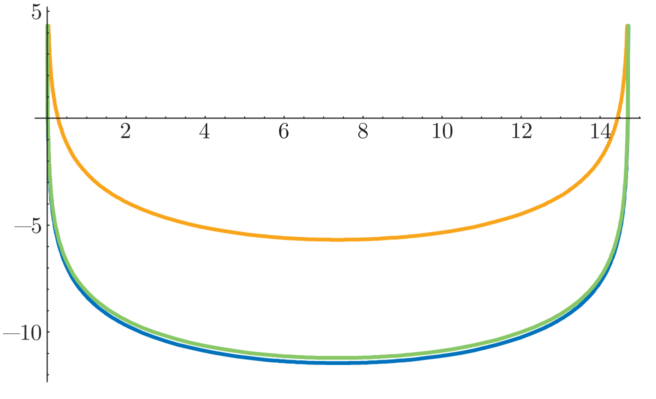

Figure 4. Graphs of functions $\gamma_1(t)$ (blue), $\gamma_2(t)$ (orange), $\gamma_3(t)$ (green) on a logarithmic scale

4. Conclusion

In this way, the formulas obtained by us can be used to solve several problems at once. Firstly, it is the task of restoring the exact form of the distribution of velocities in the "tongue" of freshwater at the outlet from the Strait of Baltiysk, at a known maximum value of the freshwater circulation at the mouth of the strait. It is worth noting that this method does not require complex field works, but only full-scale data provided by satellite surveillance systems of the LandSat-8 type.

The second important task is to control the vortex formation during the inflow of water of the Vistula Lagoon into the Baltic Sea. As is known, one of the most important tasks of ecological monitoring of seas is the location of the anthropogenic and biogenic pollution of the marine environment, and control of its propagation. The water area of the Vistula Lagoon is one of the most eutrophicated regions of the Baltic Sea, characterized by sharp seasonal fluctuations in the biomass of phytoplankton, (blue-green algae, cyanobacteria, *etc.*). Eutrophication leads to a rapid depletion of the resource of the ecosystem of the water area, a sharp decrease in the saturation of its waters with oxygen and supersaturation by dead organics. The situation is complicated by the fact that the water area of the Vistula Lagoon has an outlet to the Baltic Sea through which contaminated biogenic water factors spread across the coastal waters of the greater part of the Sambia Peninsula [11]. At the same time, the key mechanism for such a wide spread of pollution are the vortices that arise in the mouth of the Strait of Baltiysk. Thus, the problem of controlling the spread of the waters of the Vistula Lagoon in the coastal zone of the Baltic Sea turns out to be directly related to the objective of reducing the circulation of water masses – primarily, in the vicinity of the Strait of Baltiysk. As we have seen, this task can be solved by changing the velocity profile of the flow passing through the strait, for which there are a number of methods, such as changing the topography of the strait bottom, installing breakwaters running parallel to the shoreline, *etc.* Once again, we emphasize that the result of these works should be a radical (more than 2 orders!) change in the pattern of the circulation of contaminated water masses, which in turn should have the most beneficial effect on the biosystem of the entire Baltic coast of the Kaliningrad region.

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